Continuum Damage Mechanics and the Life-Fraction Rule

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Abstract

This paper gives a short review of two different methods for life prediction at high temperature; namely continuum damage mechanics (CDM) and the linear life-fraction rule (LFR). It is well-known that the class of CDM-theories with a separable evolution law gives a life prediction in accordance with the LFR. However, it appears to be an open question if this is a necessary condition. It is here shown that in order for a CDM-theory to comply with the LFR it must have a separable evolution law. That is, if we can assume that a material follows the LFR, it is necessary to chose a separable evolution law for this material. The reverse is also true, to get a life-fraction different from unity, we must chose a non-separable evolution law.

Key words: Continuum damage mechanics, Life-fraction rule, Constitutive equations, Material modelling, Deterioration

1 Introduction

At temperatures above about one third of their homologous temperature, $T_m$, engineering materials tend to rupture when exposed to a constant stress, cf. e.g. [1]. This is known as creep rupture and has long been a subject of intense research due to its engineering importance. At relatively low temperatures and high stress, rupture is preceded by large strains while at high temperature and low stress, very small strain may precede rupture.

Experiments performed at constant temperature and stress often show an exponential relation between the time to rupture, $t_R$, and the applied uniaxial stress $\sigma$. Thus,

$$t_R(\sigma) = A\sigma^a$$

(1)

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where $A$ and $a$ are material constants that can be determined experimentally.

In the sequel, the symbol $t_R(\sigma)$ is reserved for the time to creep rupture under constant stress. This function may be determined experimentally or theoretically.

In this paper, two alternative methods to predict life under varying stress are treated; the linear Life-Fraction Rule (LFR) and the method of Continuum Damage Mechanics (CDM). Specifically, we will derive the necessary and sufficient conditions for a CDM-theory to yield the same predictions as the LFR. Our interest in this problem stems from recent results by Alfredsson and Stigh, [2], who show that it is possible to develop thermodynamically consistent CDM-theories which yield a linear life-fraction different from unity. In this introductory section, the two sets of methods/theories are first presented. In the following section, some known results on the sufficient conditions under which a CDM-theory conform with the LFR are presented. In the second section, a simplified loading sequence is first presented and the necessary condition under which a CDM-theory yields the same predictions as the LFR is derived. This result is shown to be valid under general loading. The paper is ended with a discussion on the implications of this result to other situations.

1.1 Robinson’s life-fraction rule

In the industry, the most used method for life prediction is Robinson’s linear life-fraction rule, [3]. According to this method, ”the expenditure of each particular fraction of the life span at elevated temperature is independent of and without influence upon the expenditure of all other fractions of the life to rupture.” With stepwise constant stress levels, $\sigma_i$, the life-fraction is defined by

$$L_f \equiv \sum_{i=1}^{n} \frac{t_i}{t_R(\sigma_i)}.$$  \hspace{1cm} (2)

Where $t_i$ is the time spent under stress $\sigma_i$. At rupture, $L_f = 1$, which is the rupture criteria. With a continuously varying stress, $\sigma(t)$, the life-fraction is generalized to

$$L_f(t) \equiv \int_{0}^{t} \frac{dt}{t_R(\sigma(t))}.$$  \hspace{1cm} (3)

The time to rupture $t^*$, is solved from the rupture criteria $L_f(t^*) = 1$. Note that the symbol $t^*$ is reserved for the time to rupture under general loading $\sigma(t)$ while the symbol $t_R$ indicates the time to rupture under constant stress. Thus, $t_R(\sigma_0) = t^*(\sigma \equiv \sigma_0)$, where $\sigma_0$ is constant. It is immediately recognized
that $L_f(t_R) = 1$ if $\sigma = \text{const}$. An important question is obviously how good these relations conform with experiments. Due to large scatter in experimental results, this appears to be an open question which might lack a materially independent answer. An attempt to study the problem is reported by Jansson, [4]. In the study, experiments are performed on an austenitic stainless steel (AISI 316) at 700°C. Both experiments with a constant stress, yielding the relation $t_R(\sigma)$, and step-loading are performed. In these, the stress is held constant during one time period and then changed to another constant level which is kept until rupture, cf. Fig. 1. Two types of experiments are conducted, "step-up" where the second stress level is higher than the first and "step-down" where it is lower. The life-fraction varies according to Fig. 1. Thus, step-up appears to give a somewhat lower life-fraction at rupture than step-down. The difference is perhaps too small to be of any engineering significance. This is indeed the basic assumption in much engineering work.

It should be emphasized that the LFR do not couple the gradual degradation of the material with other constitutive properties. This can lead to large errors when analyzing a hyperstatic structure. If, for instance, the applied load on a hyperstatic structure is kept constant, the degradation leads to a redistribution of the stress with time. Generally, the larger stresses decrease and the lower increase. This means that an ignorant application of a design rule according to Eq. (1) gives an overly conservative estimate of the failure time.
1.2 Continuum Damage Mechanics

A method to consider the degradation of constitutive properties, integrated with a theory for life prediction is provided by the method of continuum damage mechanics (CDM). The original ideas of Kachanov, [5], is inspired by the observation that, at temperatures above about $T_m/2$, grain boundary cavities form with time. The cavities nucleate, grow and eventually coalesce to form a major crack which lead to final creep rupture. In order to consider the influence of the grain boundary cavities, a "net stress", $\tilde{\sigma}$, and a damage variable, $\omega$, are introduced; for a virgin material, $\omega = 0$ and for rupture $\omega = 1$. The net stress, which might be visualized as the load on the remaining "load bearing cross section" of the material, is related to the stress by

$$\tilde{\sigma} = \frac{\sigma}{1 - \omega}. \quad (4)$$

Thus, for a virgin material without cavities, the net stress equals the acting stress. At rupture no load bearing area remains and the net stress tends to infinity when $\omega \rightarrow 1$. Kachanov assumes the damage growth law

$$\dot{\omega} = C \left( \frac{\sigma}{1 - \omega} \right)^\nu, \quad (5)$$

where $C$ and $\nu$ are material constants. With a known $\sigma(t)$, Eq. (5) provides a separable differential equation for $\omega(t)$. With the initial condition $\omega(0) = 0$ and the rupture criteria $\omega(t^*) = 1$, integration yields

$$C(1 + \nu) \int_0^{t^*} \sigma(t)^\nu dt = 1 \quad (6)$$

This equation provides an implicit expression for the time to rupture $t^*$ under varying stress. If the stress is constant, the remaining integral is readily evaluated and the time to rupture is given by

$$t_R = \frac{1}{C(\nu + 1)\sigma^\nu}. \quad (7)$$

Thus, the theory predicts an exponential relation between $t_R$ and $\sigma$ which is indeed often observed, cf. Eq. (1). From this equation, the material constants $C$ and $\nu$ can be determined from experiments performed with constant stress.
1.3 Conditions under which CDM implies the LFR

If we assume that a material follows Kachanov’s equation, Eq. (5), it also follows the LFR. To show this, form the life-fraction at rupture using the time to rupture under constant stress from Kachanov’s theory, cf. Eq. (7). The result is

\[ L_t = C(\nu + 1) \int_0^{t^*} \sigma(t)\nu \, dt = 1, \]  

where Eq. (6) is used in the last equality. Thus, Kachanov’s version of CDM implies the LFR. This result was shown in [6] and [7]. It is also known that any separable evolution law implies the LFR. That is, if

\[ \dot{\omega} = g(\sigma)h(\omega), \]  

then \( L_t(t^*) = 1 \). This is shown by integrating the separable differential equation between \( \omega(0) = \omega_1 \) and \( \omega(t^*) = \omega_2 \), where we have introduced generalized initial and rupture conditions \( \omega(0) = \omega_1 \) and \( \omega(t^*) = \omega_2 \), respectively. The result is

\[ \int_0^{t^*} g(\sigma(t)) \, dt = \int_{\omega_1}^{\omega_2} \frac{d\omega}{h(\omega)} \equiv \Psi, \]  

where the symbol \( \Psi \) is introduced for the integral on the r.h.s. Note that with a specific CDM-model, \( \Psi \) is a constant. Now, if \( \sigma \) is a constant, the l.h.s. is readily integrated and the time to rupture is given by

\[ t_R(\sigma) = \frac{\Psi}{g(\sigma)}. \]  

With a varying stress, the life-fraction at rupture is given by Eq. (3) using Eq. (11), viz.

\[ L_t(t^*) = \int_0^{t^*} \frac{g(\sigma(t))}{\Psi} \, dt = 1, \]  

where Eq. (10) is used in the last equality.

Thus, any CDM-theory with a separable evolution law yields a life-fraction equal to unity at rupture. However, in a hyperstatic structure, any CDM-theory will generally predict another life time than the LFR since the LFR does not provide a modelling of the load redistribution due to the degradation of the material. This phenomenon is usually modelled with a CDM-theory.
2 Necessary conditions

In the introduction we showed that any evolution law

$$\dot{\omega} = f(\sigma, \omega), \quad (13)$$

that is separable according to Eq. (9) yields $L_f(t^*) = 1$ at rupture. We will now show that this is a necessary condition for a CDM-theory to conform with the LFR. To show this, first study a simplified two-step load history, cf. Fig 2. For this load history, we will show that Eq. (13) yields $L_f(t^*) = 1$ at rupture only if the function $f$ is separable according to Eq. (9). Thus, if a CDM-theory shall conform with the LFR for any load history, the only possibility is for $f$ to be separable according to Eq. (9).

For greater generality, chose $\omega(0) = \omega_1$ as the initial value of damage and $\omega(t^*) = \omega_2$ as the rupture criteria. During the time period $0 < t < T$, $\sigma = \sigma_1$ and the evolution law, Eq. (13), is separable. Integration yields

$$\int_{\omega_1}^{\omega_T} \frac{d\omega}{f(\sigma_1, \omega)} = T, \quad (14)$$

where $\omega_T$ is the damage at $t = T$. This equation is now conveniently rewritten according to

$$T = \int_{\omega_1}^{\omega_T} \ldots - \int_{\omega_1}^{\omega_2} \ldots = t_R(\sigma_1) - \int_{\omega_T}^{\omega_2} \frac{d\omega}{f(\sigma_1, \omega)}. \quad (15)$$

Similarly, in the remaining life, i.e. in the time interval $T < t \leq t^*$, $\sigma = \sigma_2$ and Eq. (13) is again a separable differential equation. Integration yields

$$\int_{\omega_T}^{\omega_2} \frac{d\omega}{f(\sigma_2, \omega)} = t^* - T. \quad (16)$$
We can now form the life-fraction at rupture, cf. Eq. (2).

\[ L_f(t^*) = \frac{T}{t_R(\sigma_1)} + t^* - T = \frac{t_R(\sigma_1) - f_{\omega_T}^{\omega_2} \frac{d\hat{\omega}}{f(\sigma_1, \sigma_2)}}{t_R(\sigma_1)} + \frac{f_{\omega_T}^{\omega_2} \frac{d\hat{\omega}}{f(\sigma_2, \sigma_2)}}{t_R(\sigma_2)}. \]  

(17)

Where Eq.'s (15) and (16) are used in the last equality. Finally, we evaluate the rupture times for the load levels \( \sigma_1 \) and \( \sigma_2 \) using Eq.'s (14) and (16). In Eq. (14) we replace \( \omega_T \) with \( \omega_1 \) and \( T \) with \( t_R(\sigma_1) \) and in Eq. (16) we replace \( \omega_T \) with \( \omega_1 \) and \( t^* - T \) with \( t_R(\sigma_2) \). Inserting in Eq. (17) and simplifying the result yields

\[ L_f(t^*) = 1 - \frac{F(\omega_T, \sigma_1)}{F(\omega_1, \sigma_1)} + \frac{F(\omega_T, \sigma_2)}{F(\omega_1, \sigma_2)}, \]  

(18)

where we have defined

\[ F(\omega, \sigma) \equiv \int_{\omega}^{\omega_2} \frac{d\hat{\omega}}{f(\sigma, \hat{\omega})}. \]  

(19)

Thus, a necessary condition for \( L_f(t^*) = 1 \) is that the relation

\[ \frac{F(\omega, \sigma)}{F(\omega_1, \sigma)} \equiv C, \]  

(20)

is independent of the parameter \( \sigma \). As evident from the equation above, the function \( C \) is, at most, a function of \( \omega \) and \( \sigma \). If \( C \) is independent of \( \sigma \) we can rewrite Eq. (20) as

\[ F(\omega, \sigma) = C(\omega)F(\omega_1, \sigma). \]  

(21)

Thus,

\[ C'(\omega)F(\omega_1, \sigma) = \int_{\omega}^{\omega_2} \frac{d\hat{\omega}}{f(\sigma, \hat{\omega})}, \]  

(22)

where we have used Eq. (19) in the last equality. Differentiation with respect to \( \omega \) yields

\[ C'(\omega)F(\omega_1, \sigma) = -\frac{1}{f(\sigma, \omega)}. \]  

(23)

Thus,

\[ f(\sigma, \omega) = g(\sigma)h(\omega), \]  

(24)

is a necessary condition for a CDM-theory to predict the same life as the LFR for a step-loading. Since a separable evolution law always implies the LFR, we
have shown that this is the only possibility for general loading.

3 Discussion

Continuum Damage Mechanics (CDM) provides methods to develop constitutive equations within the concept of continuum mechanics to deal with the gradual degradation of materials during loading. During recent years, the implications of the second law of thermodynamics on CDM-theories have been scrutinized to give guidance on the structure of a CDM-theory. A recent example is given in [2] that gives a framework for a class of CDM-theories. The basic idea is that, if the damage variable is constant, the material responds as the undamaged material with modified stress-like variables. The modifications are, in principle, given by the structure $\tilde{\sigma} = \sigma / N(\omega)$. By identifying the "damage stress", $\Omega$, as the work-conjugated quantity to $\omega$, the requirements of the evolution law for damage to conform with the dissipation inequality are identified. It is shown that this stress is partitioned in two parts; one is the elastoplastic damage stress, $\Omega^{ep}$, which essentially measures the release of elastic and plastic free energy during growth of damage. The second part is identified as a "cohesive damage stress", $\Omega^c$, which measures the increase of cohesive energy during growth of damage. This second term facilitates a possibility to model healing of damage without violating the second law of thermodynamics. In [2], the theory is exemplified with the following evolution law

$$\dot{\omega} = \frac{\sigma^2}{2E^2(1 - \omega)^2} - \frac{E_c}{E}\omega,$$  

(25)

which is both thermodynamically consistent and non-separable. In the equation, $E$ are $E_c$ are material parameters. As expected for a non-separable evolution law, $L_f(t^*) \neq 1$. Moreover, as intuitively expected the life-fraction evolves quicker than the damage, i.e. $L_f(t^*) > 1$. In order to develop a model yielding $L_f(t^*) < 1$ one may couple a time-independent damage growth to the time-dependent part. This would result in an additional increase of damage for each load cycle that yields a quicker damage growth than expected from the LFR.

The result of this paper is readily extended to fatigue if the evolution law of fatigue damage can be written

$$\frac{d\omega}{dN} = f\left(\frac{\Delta\sigma}{N(\omega)}\right).$$  

(26)

Here $\Delta\sigma$ is the stress range and $d\omega/dN$ the rate of damage growth per unit load cycle $N$. The corresponding life-fraction rule is the well known Palmgren-
Miner rule, cf. [8] and [9]

\[
\sum_{0}^{N_F} \frac{n_i}{N_f(\Delta \sigma_i)} = 1.
\]

(27)

Here, \(n_i\) is the number of load cycles at the constant stress range \(\Delta \sigma_i\), \(N_f(\Delta \sigma_i)\) is the number of load cycles to fracture at the same stress range and \(N_F\) is the number of load blocks to fracture. A simple change of interpretation of the symbols leads to the same conclusion as for creep rupture.

We conclude this paper by noting that the result is valid for multiaxial stress states as well. For instance, introducing an effective stress according to

\[
\sigma = \alpha \sigma_{\text{eff}} + 3(1-\alpha)\sigma_m.
\]

(28)

where \(\sigma_{\text{eff}} \equiv (3/2)s_{ij}s_{ij}\), \(\sigma_m \equiv \sigma_{kk}/3\) and \(s_{ij} \equiv \sigma_{ij} - \delta_{ij}\sigma_m\) are the von Mises effective stress, the mean stress and the deviatoric stress respectively. With \(0 \leq \alpha \leq 1\), the damage evolution can be modelled to vary from one being driven by the hydrostatic stress (\(\alpha = 0\)) to one being driven by the deviatoric stress (\(\alpha = 1\)). With this effective stress, uniaxial tests can be used to determine the evolution law \(f(\sigma, \omega)\). The value of \(\alpha\) have to be determined from multiaxial tests. With this effective stress, the results of this paper is immediately applicable. With more elaborate evolution laws, the ability to conform with the LFR has to be checked. If possible, a suitable method would be to identify the effective stress associated with the specific model.

References


